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A Noncommutative Korovkin Theorem

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THEOREM. *Let $\mathscr{A}$ be a C^*-algebra with unit I, and let $\{P_n\}$ be a sequence of positivity-preserving linear maps sending $\mathscr{A}$ into $\mathscr{A}$ and satisfying $P_n I \leqslant I$. Then*

$$\mathscr{J} = \{T \in \mathscr{A} \mid T = T^*, P_n T \to T, P_n T^2 \to T^2\}$$

is a norm-closed Jordan algebra of self-adjoint elements of $\mathscr{A}$, i.e., a real linear subspace of $\mathscr{A}$ closed under the Jordan product $T \circ L = (TL + LT)/2$.

When $\mathscr{A} = C[a, b]$, this reduces to a version of Korovkin's well-known theorem on approximation by positivity-preserving linear methods. Analogs of the theorem above are shown to hold in the weak operator topology and the strong operator topology. Also considered is convergence in trace norm when $\{P_n\}$ acts on the trace class.

The only type of approximation procedure considered here is that using *positivity-preserving* linear methods, which (see [3]) plays an important role in commutative approximation theory. The well-known theorem of Korovkin asserts that if $\{P_n\}$ is a sequence of positivity-preserving linear maps of $C[a, b]$ into itself, then $P_n f \to f$ for all f in $C[a, b]$, provided only that $P_n 1 \to 1$, $P_n x \to x$, and $P_n x^2 \to x^2$. In other words, if the (positivity-preserving and linear) approximation procedure works for x and x^2, and for 1 (and 1^2), then it works for the Banach algebra generated by x and 1.

This leads us to study the thesis: "Success by an approximation method on some elements *and their squares* implies success on the algebra generated by those elements." We shall see that, under mild hypotheses, this thesis holds in algebras of operators (with diverse notions of convergence), provided we consider the algebra generated using the *Jordan product*

$$T \circ L = (TL + LT)/2.$$

In the noncommutative situations in which we are interested, the Jordan product is commutative, distributive, but not associative.

Korovkin's theorem has been generalized already to various Banach space settings. Some of the latest work may be found in [5]. Our approach is along the lines of [1], which seems to be the earliest attempt to consider Korovkin's theorem in the setting of a Banach algebra.

Throughout, $\{P_n\}$ will denote a sequence of positivity-preserving linear maps from the algebra under consideration into itself. We first consider norm convergence when $\{P_n\}$ acts on a C^*-algebra. In later sections we consider weaker and stronger types of convergence.

I. Norm Convergence in C^*-Algebras

A C^*-algebra is a (complex) Banach algebra $\mathscr{A}$ with involution $*$ whose norm satisfies $\| T^*T \| = \| T \|^2$ for T in $\mathscr{A}$. By the theorem of Gelfand and Naimark [8, Theorem 1.16.6, p. 41] a C^*-algebra is $*$-isomorphic to a norm-closed self-adjoint subalgebra of $\mathscr{L}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. When $\mathscr{A}$ is identified with such a subalgebra, then a *positive* element of $\mathscr{A}$ is an operator T on $\mathscr{H}$ that is self-adjoint ($T = T^*$) and whose spectrum $\operatorname{sp}(T)$ is contained in the nonnegative reals. We write $T \geqslant 0$ in this case. We say that a linear map $P: \mathscr{A} \to \mathscr{A}$ is *positivity-preserving* if $T \geqslant 0$ implies $PT \geqslant 0$.

Let $\mathscr{A}$ be a C^*-algebra with unit I, and let $\mathscr{A}_s$ denote the set of all self-adjoint (s.a.) elements of $\mathscr{A}$. Let $\{P_n\}$ act on $\mathscr{A}$. Assume

$$P_n I \leqslant I. \tag{1.1}$$

We are interested in the following sets:

$$\begin{aligned} \mathscr{U} &= \{T \in \mathscr{A}_s \mid \| P_n T - T \| \to 0\}, \\ \mathscr{J} &= \{T \in \mathscr{U} \mid T^2 \in \mathscr{U}\}. \end{aligned}$$

The idea of the following proof will be used again in the sequel.

Theorem 1.1. *$\mathscr{J}$ is a Jordan algebra of s.a. elements of $\mathscr{A}$, i.e., $\mathscr{J}$ is a real linear subspace of $\mathscr{A}_s$, closed under the Jordan product.*

Proof. $\mathscr{J}$ is clearly closed under multiplication by real scalars. We need only show $\mathscr{J}$ closed under *squares* and *sums*, since $2T \circ L = (T + L)^2 - T^2 - L^2$.

Assume $T \in \mathscr{A}_s$, $L \in \mathscr{A}_s$, $\lambda \in \mathbf{R}$. As we go on, more will be assumed of T and L. Kadison's Schwarz inequality [4] asserts that if $P: \mathscr{A} \to \mathscr{A}$ is linear, positivity-preserving, and $PI \leqslant I$, then

$$P(X)^2 \leqslant P(X^2), \qquad X \in \mathscr{A}_s. \tag{1.2}$$

Letting $X = T + \lambda L$ results in

$$P_n(T)^2 + 2\lambda P_n(T) \circ P_n(L) + \lambda^2 P_n(L)^2 \\ \leqslant P_n(T^2) + 2\lambda P_n(T \circ L) + \lambda^2 P_n(L^2). \tag{1.3}$$

Assume $\| L \| \leqslant 1$, so that $L^2 \leqslant \| L^2 \| I = \| L \|^2 I \leqslant I$. Then,

$$P_n(L^2) - P_n(L)^2 \leqslant P_n(L^2) \leqslant P_n(I) \leqslant I. \tag{1.4}$$

Relations (1.3) and (1.4) imply that for all real λ,

$$2\lambda P_n(T) \circ P_n(L) - 2\lambda P_n(T \circ L) \leqslant P_n(T^2) - P_n(T)^2 + \lambda^2 I. \tag{1.5}$$

Now, assume $T \in \mathscr{J}$. Then, $P_n(T^2) - P_n(T)^2 \to 0$. Given $\epsilon > 0$, find N such that $n \geqslant N$ implies $\| P_n(T^2) - P_n(T)^2 \| \leqslant \epsilon^2$. Then, by (1.5), $n \geqslant N$ implies

$$2\lambda[P_n(T) \circ P_n(L) - P_n(T \circ L)] \leqslant \epsilon^2 I + \lambda^2 I,$$

where N depends on ϵ but not on λ. $\lambda = \pm\epsilon$ yields

$$-\epsilon \leqslant P_n(T) \circ P_n(L) - P_n(T \circ L) \leqslant \epsilon, \qquad n \geqslant N. \tag{1.6}$$

Finally, assume $L \in \mathscr{U}$. (1.6) then shows $\limsup \| T \circ L - P_n(T \circ L)\| \leqslant \epsilon$. Hence, $T \circ L \in \mathscr{U}$. The assumption $\| L \| \leqslant 1$ being inessential, we have proved that

$$T \in \mathscr{J} \qquad \text{and} \qquad L \in \mathscr{U} \ \text{ implies } \ T \circ L \in \mathscr{U}. \tag{1.7}$$

By induction using (1.7), $T \in \mathscr{J}$ implies $T^k \in \mathscr{U}$. In particular, $T \in \mathscr{J}$ implies $T^2 \in \mathscr{J}$, so $\mathscr{J}$ is closed under squares. To see that $\mathscr{J}$ is closed under sums, assume $\mathscr{J}$ contains T_1 and T_2. Then $\mathscr{U}$, being a real subspace, contains $T_1 + T_2$, as well as T_1 and T_2. The subspace $\mathscr{U}$ also contains

$$(T_1 + T_2)^2 = T_1 \circ (T_1 + T_2) + T_2 \circ (T_1 + T_2),$$

since each of these terms is in $\mathscr{U}$ by (1.7). Hence, $\mathscr{J}$ contains $T_1 + T_2$. This completes the proof.

LEMMA 1.2. *$\mathscr{U}$ and $\mathscr{J}$ are norm-closed.*

Proof. If T is in $\mathscr{A}_s$, then $-\| T \| \leqslant T \leqslant \| T \|$, so $P_n I \leqslant I$ implies $\| P_n T \| \leqslant \| T \|$. Using this, a standard 3ϵ-argument shows $\mathscr{U}$ to be norm-closed, from which it follows trivially that $\mathscr{J}$ is norm-closed.

Consider a single s.a. operator T. The foregoing shows that the approximation method works on the C^*-algebra generated by T, provided only that it works on T and on T^2. A second consequence is that $\mathscr{J} + i\mathscr{J}$ is a "J^*-algebra," i.e., a norm-closed, *-closed, complex-linear subspace of $\mathscr{A}$ that is closed under the Jordan product. This makes our Main Theorem easy to prove.

THEOREM 1.3. *Let $\{P_n\}$ be a sequence of positivity-preserving linear maps of a C^*-algebra $\mathscr{A}$ into itself such that $P_n I \leqslant I$. Then,*

(a) $\mathscr{J} = \{T \in \mathscr{A}_s \mid P_n T \to T, P_n T^2 \to T^2\}$ *is a norm-closed Jordan algebra of s.a. elements of* $\mathscr{A}$;

(b) $\mathscr{C} = \{T \in \mathscr{A} \mid P_n T \to T, \ P_n T^2 \to T^2, \ P_n T^* \circ T \to T^* \circ T\}$ *is a J^*-algebra.*

Proof. It remains only to show $\mathscr{C} = \mathscr{J} + i\mathscr{J}$. The remark made above implies $\mathscr{J} + i\mathscr{J} \subset \mathscr{C}$. To see the reverse inclusion, assume $C \in \mathscr{C}$. P_n, being complex-linear and positivity-preserving, commutes with involution. Thus, $P_n(C^*) \to C^*$ (implying that $T \equiv \operatorname{Re} C$ and $L \equiv \operatorname{Im} C$ are in $\mathscr{U}$) and $P_n(C^{*2}) \to C^{*2}$. The subspace $\mathscr{U}$ contains T^2, since

$$4T^2 = C^2 + C^{*2} + 2C^* \circ C;$$

and $\mathscr{U}$ contains $C^* \circ C - T^2 = L^2$ as well. Hence, $T \in \mathscr{J}$ and $L \in \mathscr{J}$, i.e., $C \in \mathscr{J} + i\mathscr{J}$. This completes the proof.

Instead of (1.1), we could have assumed $\| P_n I \| \to 1$, as can be seen by applying Theorem 1.1 to $P_n' = (1/\| P_n I \|) P_n$ and observing $P_n' T \to T$ if and only if $P_n T \to T$.

$\mathscr{C}$ need not be an algebra (unless $\mathscr{A}$ is commutative): Let $\mathscr{A}$ be the algebra of 2×2 matrices with complex entries, let $P: \mathscr{A} \to \mathscr{A}$ be the transpose operator, and let $\{P_n\}$ be the constant sequence $P_n = P$. Then, $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $L = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ belong to $\mathscr{C}$, but TL does not.

When $\mathscr{A}$ is commutative, the Jordan product becomes the ordinary product, a J^*-algebra becomes a C^*-algebra, and a Jordan algebra of s.a. elements can be identified with an algebra of real functions. Let X be a compact Hausdorff space, let $C(X)$ and $C_{\mathrm{C}}(X)$ denote, respectively, the algebras of real-valued and complex-valued continuous functions on X, with supremum norm. Taking $\mathscr{A} = C_{\mathrm{C}}(X)$, we can write the commutative version of Theorem 1.3 in more familiar notation.

THEOREM 1.4. *Let $\{P_n\}$ act on $C_{\mathrm{C}}(X)$. Assume $P_n 1 \leqslant 1$ (or $\| P_n 1 \| \to 1$). Then,*

(a) $\{f \in C(X) \mid P_n f \to f, P_n f^2 \to f^2\}$ *is a real C^*-algebra;*

(b) $\{f \in C_{\mathrm{C}}(X) \mid P_n f \to f, P_n f^2 \to f^2, P_n |f|^2 \to |f|^2\}$ *is a complex C^*-algebra.*

When $X = [a, b]$, Korovkin's theorem is a corollary of (a), as a consequence of the Weierstrass polynomial approximation theorem.

Theorem 1.4, in conjunction with the Stone–Weierstrass theorem, could be useful in proving the effectiveness of an approximation method on $C(X)$. To employ the noncommutative result, a "Jordanized" Stone–Weierstrass theorem might be helpful.

II. Weak and Strong Convergence

Let $\mathscr{H}$ be a complex Hilbert space with inner product $\langle\cdot, \cdot\rangle$, let $\mathscr{L}(\mathscr{H})$ be the C^*-algebra of all bounded linear maps of $\mathscr{H}$ into $\mathscr{H}$, and let $\mathscr{A}$ be a $*$-closed algebra contained in $L(\mathscr{H})$ and containing I. Assume P_n (positivity-preserving) sends $\mathscr{A}$ into $\mathscr{A}$ and satisfies $P_n I \leqslant I$. Since P_n extends naturally to the norm-closure of $\mathscr{A}$, a C^*-algebra, Kadison's inequality holds on P_n. We are interested in the same thing as before, but with weaker notions of convergence. For a sequence $\{T_n\}$ in $\mathscr{A}$ we write $T_n \to T\,(\mathrm{wk})$, or $T_n \to T\,(\mathrm{st})$, to denote, respectively, convergence of $\{T_n\}$ to T in the weak, or strong, operator topology.

Theorem 2.1. *$\mathscr{J}_{\mathrm{wk}} = \{T \in \mathscr{A}_s \mid P_n T \to T\,(\mathrm{wk}), P_n T^2 \to T^2\,(\mathrm{wk})\}$ and $\mathscr{J}_{\mathrm{st}} = \{T \in \mathscr{A}_s \mid P_n T \to T\,(\mathrm{st}), P_n T^2 \to T^2\,(\mathrm{st})\}$ are Jordan algebras of s.a. operators. Moreover, $\mathscr{J}_{\mathrm{wk}} = \mathscr{J}_{\mathrm{st}}$.*

Proof. Since most of the proof is like that of Theorem 1.1, we simply indicate the necessary modifications. We first consider weak convergence. Let φ be a unit vector in $\mathscr{H}$. From (1.5) we have $T \in \mathscr{A}_s$, $L \in \mathscr{A}_s$, $\| L \| \leqslant 1$ implies that for all real λ,

$$2\lambda\langle(P_n T \circ P_n L - P_n T \circ L)\varphi, \varphi\rangle \leqslant \langle(P_n(T^2) - P_n(T)^2)\varphi, \varphi\rangle + \lambda^2.$$

In order that the argument be allowed to proceed as before, we must show that $T \in \mathscr{J}_{\mathrm{wk}}$ implies the inner product on the right tends to zero.

A slight difficulty arises here. We need to know that $P_n(T)^2 \to T^2\,(\mathrm{wk})$, and this does not follow from the condition $P_n T \to T\,(\mathrm{wk})$, the squaring operation being discontinuous in the weak topology. However, assume $T \in \mathscr{J}_{\mathrm{wk}}$ and consider the expansion

$$\begin{aligned}\|(P_n T - T)\varphi\|^2 = \langle P_n(T)^2 \varphi, \varphi\rangle &- \langle T\varphi, P_n T\varphi\rangle \\ &- \langle P_n T\varphi, T\varphi\rangle + \langle T\varphi, T\varphi\rangle. \qquad (2.1)\end{aligned}$$

By (1.2), $\langle P_n(T)^2 \varphi, \varphi\rangle \leqslant \langle P_n(T^2)\varphi, \varphi\rangle$. From this and (2.1), it is easy to conclude $\limsup \|(P_n T - T)\varphi\|^2 = 0$. We have thus proved that

$$T \in \mathscr{J}_{\mathrm{wk}} \quad \text{implies} \quad P_n T \to T\,(\mathrm{st}). \qquad (2.2)$$

Now that we have strong convergence of P_nT it follows that

$$\langle P_n(T)^2 \varphi, \varphi\rangle = \langle P_nT\varphi, P_nT\varphi\rangle \to \langle T\varphi, T\varphi\rangle = \langle T^2\varphi, \varphi\rangle,$$

overcoming our difficulty and allowing the argument to proceed as before. To complete the argument we need to know that $\langle P_nT \circ P_nL\varphi, \varphi\rangle$ tends to $\langle T \circ L\varphi, \varphi\rangle$ if $T \in \mathscr{J}_{\mathrm{wk}}$ and $P_nL \to L$ (wk). This also follows easily with help from (2.2). Hence, $\mathscr{J}_{\mathrm{wk}}$ is a Jordan algebra.

To finish the proof of Theorem 2.1 we show $\mathscr{J}_{\mathrm{wk}} = \mathscr{J}_{\mathrm{st}}$. Assume $T \in \mathscr{J}_{\mathrm{wk}}$. Then $T^4 \in \mathscr{J}_{\mathrm{wk}}$ since $\mathscr{J}_{\mathrm{wk}}$ is a Jordan algebra, and hence, $\langle P_n(T^4)\varphi, \varphi\rangle$ tends to $\langle T^4\varphi, \varphi\rangle$ for all φ in $\mathscr{H}$. Using this and the argument proving (2.2) (with T^2 in place of T), we conclude $P_nT^2 \to T^2$ (st). Thus, $T \in \mathscr{J}_{\mathrm{st}}$. This completes the proof.

We refrain from stating the corollaries of Theorem 2.1 analogous to those of Theorem 1.1. Instead, we note some facts to be used in the sequel.

In case $\mathscr{A}$ does not contain I, we could have assumed instead that $\| P_nT\| \leqslant \| T\|$, if $T \in \mathscr{A}_s$. Then, by adjoining I and extending P_n linearly (leaving I fixed), we are back in the original situation.

Assume $T \in \mathscr{J}_{\mathrm{wk}}$ and $E \in \mathscr{A}$, where E is a spectral projection of T. When does it follow that $E \in \mathscr{J}_{\mathrm{wk}}$? A sufficient condition is *that E be the spectral projection associated with an open and closed subset of* $\mathrm{sp}(T)$. To see this, it is enough to consider the case when $\mathscr{A}$ is norm-closed, since we assume $E \in \mathscr{A}$ at the outset. In this case $\mathscr{J}_{\mathrm{wk}}$ easily can be shown norm-closed, so Theorem 2.1 implies $\mathscr{J}_{\mathrm{wk}}$ contains the closure of the set of polynomials in T, which contains E.

Another condition that implies $E \in \mathscr{J}_{\mathrm{wk}}$ is that E be a projection associated with an interval whose end points do not lie in the pure point spectrum of T. However, since this result is not needed in what follows and since its proof is detailed, we omit the proof.

III. Convergence in the Trace Class

Let $\{\varphi_j\}$ be an orthonormal (o.n.) basis for a Hilbert space $\mathscr{H}$. The trace class $\mathscr{T}$ consists of all operators T in $\mathscr{L}(\mathscr{H})$ such that

$$\| T\|_{\mathscr{T}} = \mathrm{tr} \mid T \mid \equiv \sum \langle \mid T \mid \varphi_j, \varphi_j\rangle < \infty,$$

where $\mid T \mid = (T^*T)^{1/2}$. $\| T\|_{\mathscr{T}}$ is independent of choice of o.n. basis, and $\mathscr{T}$, $\| \cdot \|_{\mathscr{T}}$ is a Banach algebra [9]. The collection of s.a. elements of $\mathscr{T}$ is denoted by $\mathscr{T}_s$.

We are interested in the same thing as before, but with the stronger notion of convergence in trace norm. The following lemma casts some light on convergence in the trace class.

LEMMA (Davies [2, p. 291]). *Let $\{T_n\}$ be a sequence of positive operators in $\mathscr{T}_s$ such that $T_n \to T$ (wk) and T is in $\mathscr{T}_s$. Then, $\| T - T_n \|_{\mathscr{T}} \to 0$ if and only if* $\operatorname{tr} T_n \to \operatorname{tr} T$.

Since $I \notin \mathscr{T}$, the assumptions of our theorem are formulated a little differently, the first assumption being stated in terms of the operator norm rather than the trace norm. We must also assume that the norms of the P_n, as maps of $\mathscr{T}$, $\| \cdot \|_{\mathscr{T}}$ into itself, are uniformly bounded. As usual, P_n is assumed linear and positivity-preserving.

THEOREM 3.1. *Let $\{P_n\}$ act on $\mathscr{T}$. Assume*

(i) $\| P_n T \| \leqslant \| T \|, \qquad T \in \mathscr{T}_s$;

(ii) $\sup \| P_n \| < \infty$.

Then, $\mathscr{J}_{\mathrm{tr}} = \{T \in \mathscr{T}_s \mid \| P_n T - T \|_{\mathscr{T}} \to 0, \| P_n T^2 - T^2 \|_{\mathscr{T}} \to 0\}$ is a $\| \cdot \|_{\mathscr{T}}$-closed Jordan algebra of s.a. operators in $\mathscr{T}$.

Proof. The proof is broken down into Lemma 3.2, Lemma 3.3, and Corollary 3.5.

LEMMA 3.2. *$\mathscr{J}_{\mathrm{tr}}$ is a real subspace.*

Proof. Assume T and L are in $\mathscr{J}_{\mathrm{tr}}$. Taking the trace of both sides of (1.3) and using once again the basic argument there, we conclude $\operatorname{tr} P_n(T \circ L) \to \operatorname{tr} T \circ L$. Since we have already $\operatorname{tr} P_n T^2 \to \operatorname{tr} T^2$, and $\operatorname{tr} P_n L^2 \to \operatorname{tr} L^2$, it follows that

$$\begin{aligned} \operatorname{tr} P_n(T + L)^2 = \operatorname{tr} P_n T^2 + 2 \operatorname{tr} P_n(T \circ L) \\ + \operatorname{tr} P_n L^2 \to \operatorname{tr}(T + L)^2. \end{aligned} \tag{3.1}$$

Assumption (i) justifies the application of Theorem 2.1 to this situation. Thus, since $\mathscr{J}_{\mathrm{wk}}$ is a Jordan algebra and $\mathscr{J}_{\mathrm{tr}} \subset \mathscr{J}_{\mathrm{wk}}$, we have

$$P_n(T + L)^2 \to (T + L)^2 \quad (\mathrm{wk}). \tag{3.2}$$

Since $P_n(T + L)^2 \geqslant 0$, (3.1), (3.2), and Davies' lemma show that $\| P_n(T + L)^2 - (T + L)^2 \|_{\mathscr{T}} \to 0$. Hence, $T + L$ is in $\mathscr{J}_{\mathrm{tr}}$, from which the lemma follows.

LEMMA 3.3. *$\mathscr{J}_{\mathrm{tr}}$ is $\| \cdot \|_{\mathscr{T}}$-closed.*

Proof. This is a consequence of assumption (ii) and a standard argument.

LEMMA 3.4. *Assume $T \in \mathscr{T}_s$. $\mathscr{J}_{\mathrm{tr}}$ contains T if and only if $\mathscr{J}_{\mathrm{tr}}$ contains each spectral projection of T corresponding to a nonzero eigenvalue.*

Proof. Let $\sum \lambda_i E_i$ be the spectral representation of T and assume $\mathscr{I}_{\mathrm{tr}}$ contains each E_i. By Lemma 3.2, $\mathscr{I}_{\mathrm{tr}}$ contains $\sum_1^N \lambda_i E_i$ for each positive integer N. Now, Lemma 3.3 implies $T \in \mathscr{I}_{\mathrm{tr}}$.

Conversely, assume $T \in \mathscr{I}_{\mathrm{tr}}$ and let E be a spectral projection of T whose associated eigenvalue λ is nonzero. The eigenvalue λ is then an isolated point of $\mathrm{sp}(T)$, so $P_n E \to E$ (wk) by the remarks following the proof of Theorem 2.1.

We now show $\mathrm{tr}\, P_n E \to \mathrm{tr}\, E$, in order to invoke Davies' lemma. First define $P_n I = I$ and let P_n be extended linearly to the domain $\mathscr{T} + xI$. Let

$$c = \min\{|\lambda - \lambda_i| \mid \lambda_i \in \mathrm{sp}(T), \lambda_i \neq \lambda\},$$

and let φ_i be a unit eigenvector of T with eigenvalue λ_i different from λ. Cauchy's inequality on the positive linear functional $\langle P_n(\cdot)\, \varphi_i, \varphi_i\rangle$ together with the obvious inequality $cE \leqslant E\,|T - \lambda_i|$ shows

$$c^2 \langle P_n E \varphi_i, \varphi_i\rangle^2 \leqslant \langle P_n E \varphi_i, \varphi_i\rangle \langle P_n (T - \lambda_i)^2 \varphi_i, \varphi_i\rangle.$$

Hence,

$$\begin{aligned} c^2 \langle P_n E \varphi_i, \varphi_i\rangle &\leqslant \langle P_n (T - \lambda_i)^2 \varphi_i, \varphi_i\rangle \\ &= \langle P_n (T - \lambda_i)^2 \varphi_i, \varphi_i\rangle + \langle (T - \lambda_i)^2 \varphi_i, \varphi_i\rangle \\ &= \langle (P_n T^2 - T^2)\, \varphi_i, \varphi_i\rangle - 2\lambda_i \langle (P_n T - T)\, \varphi_i, \varphi_i\rangle \\ &\leqslant \langle | P_n T^2 - T^2 |\, \varphi_i, \varphi_i\rangle + 2\, \| T \|\, \langle | P_n T - T |\, \varphi_i, \varphi_i\rangle. \end{aligned}$$

Summing this over any o.n. set $\{\varphi_i\}$ of eigenvectors whose eigenvalues differ from λ, we deduce

$$0 \leqslant c^2 \sum \langle P_n E \varphi_i, \varphi_i\rangle \leqslant \| P_n T^2 - T^2 \|_{\mathscr{T}} + 2\, \| T \|\, \| P_n T - T \|_{\mathscr{T}},$$

which tends to zero. Therefore,

$$\sum \langle (P_n E - E)\, \varphi_i, \varphi_i\rangle = \sum \langle P_n E \varphi_i, \varphi_i\rangle \to 0, \tag{3.3}$$

where $\{\varphi_i\}$ is an o.n. set spanning Range $E^\perp$.

Now, let $\{\psi_j\}$ be an o.n. basis for Range E. Since $\{\psi_j\}$ is a finite set and $P_n E \to E$ (wk),

$$\sum \langle (P_n E - E)\, \psi_j, \psi_j\rangle \to 0. \tag{3.4}$$

By (3.3) and (3.4), $\mathrm{tr}(P_n E - E) \to 0$ as required. Davies' lemma implies $\| P_n E - E \|_{\mathscr{T}} \to 0$. Since $E^2 = E$ it follows that $E \in \mathscr{I}_{\mathrm{tr}}$.

COROLLARY 3.5. *$\mathscr{I}_{\mathrm{tr}}$ is closed under squares.*

This completes the proof of Theorem 3.1. As in the preceding section, we forego the formulation of an analog of Theorem 1.3.

IV. Concluding Remarks

Theorem 1.3 was conjectured in the author's thesis [7], where it was proved for the elementary case when $\mathscr{A} = M_n$.

Theorem 1.4 is almost as old as Korovkin's theorem. It is essentially the same as Theorem 1.2 of [6], where the positivity-preserving operators are assumed to form a semigroup. Nelson's proof does not require this hypothesis, however.

Many of the results here are true when one considers algebras of operators on a *real* Hilbert space. However, conclusion (b) of Theorem 1.3 fails, as shown by this example: Let $\mathscr{A}$ be the algebra of 3×3 matrices with real entries, where transposition plays the role of involution. Define $P: \mathscr{A} \to \mathscr{A}$ by

$$P\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & z \end{pmatrix},$$

where $z = a_{33} + a_{12} - a_{21} + a_{13} - a_{31}$. Since P leaves fixed each symmetric matrix, the constant sequence $P_n = P$ satisfies the hypotheses of Theorem 1.3. Let

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then P leaves fixed $T = T^*$ and T^2, and also leaves fixed L, L^2, and $L^* \circ L$ (not to mention L^*, L^{*2}, L^*L, and LL^*), so that T and L are in $\mathscr{C}$. Nevertheless, $T \circ L \notin \mathscr{C}$. The trouble is that P does not commute with involution.

Since the Jordan product is not well known, a brief word describing its relationship to the ordinary product may be in order. Two identities are

$$TLT = 2T \circ (T \circ L) - (T \circ T) \circ L,$$

$$(TL - LT)^2/4 = (T \circ L)^2 - T \circ (L^2 \circ T) - L \circ (T^2 \circ L) + T^2 \circ L^2.$$

These show that if a Jordan algebra contains T and L, then it contains TLT and $(TL - LT)^2$. It also can be shown that if $T_1, \ldots, T_n$ are in a Jordan algebra, then so is their *symmetric product*

$$(1/n!) \sum T_{\pi(1)} \cdots T_{\pi(n)}$$

where π ranges over all permutations of n letters.

Topping [10] discusses weakly closed Jordan algebras of self-adjoint operators.

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